# CHEEGER-GROMOV CONVERGENCE IN A CONFORMAL SETTING

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Abstract. For a sequence $\{(M_i, g_i, x_i)\}$ of pointed Riemannian manifolds with boundary, the sequence $\{(M_i, \tilde{g}_i, x_i)\}$ is its conformal satellite if the metric $\tilde{g}_i$ is conformal to $g_i$, that is, $\tilde{g}_i = u_i^{\frac{4}{n-2}} g_i$. Assuming the manifolds $(M_i, g_i, x_i)$ have uniformly bounded geometry, we show that both sequences have smoothly Cheeger-Gromov convergent subsequences provided the conformal factors $u_i$ are principal eigenfunctions of an appropriate elliptic operator. Part of our result is a Cheeger-Gromov compactness for manifolds with boundary. We use stable versions of classical elliptic estimates and inequalities found in the recently established 'flatzoomer' method.



Contents

## 1. Introduction

1.1. **Motivation.** We say that a sequence $\{(M_i, g_i, x_i)\}$ of pointed Riemannian manifolds $C^k$-converges (in the sense of Cheeger-Gromov, see, say, the papers [5] and [10]) to a Riemannian manifold $(M_\infty, g_\infty, x_\infty)$ if and only if for all integers $m \geq 1$ eventually

(a) there is a diffeomorphism $\phi_i^{(m)} : B_m(x_\infty) \to B_m(x_i)$ mapping $x_\infty$ to $x_i$;

(b) the metrics $(\phi_i^{(m)})^* g_i$ converge to $g_\infty$ in the $C^k$-norm on $B_m(x_\infty)$.

Here $B_m(x)$ denotes a geodesic ball of radius $m$ centered at $x$.

In the case if there is a $C^k$-converging subsequence of a sequence $\{(M_i, g_i, x_i)\}$ converging to $(M_\infty, g_\infty, x_\infty)$, we say that a sequence $\{(M_i, g_i, x_i)\}$ *$C^k$-subconverges to* $(M_\infty, g_\infty, x_\infty)$.

Let $\{(M_i, g_i, x_i)\}$ be a sequence of pointed[1] Riemannian $n$-dimensional manifolds, with possibly non-empty boundary, such that there exists a smooth Cheeger-Gromov limit

$$(M_\infty, g_\infty, x_\infty) = \lim_{i\to\infty} (M_i, g_i, x_i),$$

where the limiting manifold might be noncompact even if all $M_i$ are compact, and have non-empty (and non-compact) boundary as well. There are many examples that provide such limits.

Assume now that each manifold $M_i$ of the above sequence is given a conformal metric $\tilde{g}_i = u_i^{\frac{4}{n-2}} g_i$. We call this a *satellite sequence* of manifolds $\{(M_i, \tilde{g}_i, x_i)\}$. We address the following natural questions:

**Question 1.** *Does the satellite sequence $\{(M_i, \tilde{g}_i, x_i)\}$ converge to a smooth manifold?*

**Question 2.** *If it does converge to a smooth manifold $(\tilde{M}_\infty, \tilde{g}_\infty, \tilde{x}_\infty)$, what is a relationship between the limiting manifolds $(M_\infty, g_\infty, x_\infty)$ and $(\tilde{M}_\infty, \tilde{g}_\infty, \tilde{x}_\infty)$? In particular, when are the manifolds $M_\infty$ and $\tilde{M}_\infty$ diffeomorphic and the metrics $g_\infty$ and $\tilde{g}_\infty$ conformal?*

If we assume uniform bounds on the geometry of the manifolds $(M_i, g_i, x_i)$, then the limiting manifold (which is, in general, noncompact) has bounded geometry, as defined below. However, it is easy to construct examples when the satellite sequence $\{(M_i, \tilde{g}_i, x_i)\}$ fails to have uniformly bounded geometry, and hence this sequence might fail to converge. Even if the satellite sequence $\{(M_i, \tilde{g}_i, x_i)\}$ does converge, its limit a priori have no obvious relation to the limit of the original sequence.

In a recent paper [11], the second author and Marc Nardmann introduced the "flatzoomer" method. This technique worked efficiently to show that any non-compact Riemannian manifold could be conformally modified to get a metric of bounded geometry. In this article, we show that the estimates involved in the "flatzoomer" method can also be used to control the geometry of the limits of the satellite sequences $\{(M_i, \tilde{g}_i, x_i)\}$ under Cheeger-Gromov smooth convergence. Furthermore, we use stable versions of elliptic inequalities to give answers to the above questions in the case when the conformal functions $u_i$ are positive solutions of relevant elliptic problems.

[1] We follow the convention that a pointed manifold is a manifold with a base point in every connected component.

1.2. **Bounded geometry.** For a Riemannian metric $h$, we denote by $\mathrm{Rm}_h$ its Riemannian tensor, and by $\mathrm{inj}_h$ its injectivity radius. Let $(M, g, x)$ be a pointed Riemannian manifold. In the case $M$ has has non-empty boundary $\partial M$, we denote by $\partial g = g|_{\partial M}$ the induced metric. Denote by $d$ the distance function induced by the metric $g$. Then for given $r > 0$ we denote by $B_r(\partial M)$ a tubular neighborhood of $\partial M$ of radius $r$, i.e., $B_r(\partial M) = \{ x \in M \mid d(x, \partial M) < r \}$. In the following, we adopt the following definition of bounded geometry for manifolds with boundary (cf., e.g., [13]):

**Definition 1.1.** Fix a positive integer $k$ and a constant $c > 0$. A Riemannian manifold $(M, g)$ with non-empty boundary $\partial M$ has $(c, k)$-*bounded geometry* if

(i) for the inward normal vector field $\nu$, the normal exponential map $E : \partial M \times [0, c^{-1}] \to M$, $E(y, r) := \exp_y(r\nu)$, is a diffeomorphism onto its image;

(ii) $\mathrm{inj}_{\partial g}(\partial M) \geq c^{-1}$;

(iii) $\mathrm{inj}_g(M \setminus B_r(\partial M)) \geq r$ for all $r \leq c^{-1}$;

(iv) $|\nabla^l_g \mathrm{Rm}_g|_g \leq c$ and $|\nabla^l_{\partial g} \mathrm{Rm}_g|_g \leq c$ for all $l \leq k$.

For a pointed Riemannian manifold $(M, g, x)$ we moreover require that for the basepoint $x_i$ of every connected component $M_i$ we have $d(x_i, \partial M_i) \geq 2c^{-1}$.

**Remark 1.2.** It is known that the above requirements guarantee that the boundary manifold $(\partial M, \partial g)$ also has $(c, k)$-bounded geometry, see [13]. In the case when $\partial M = \emptyset$, some of requirements are empty, and the condition (iii) is the same as $\mathrm{inj}_g \geq c^{-1}$.

1.3. **Conformal Laplacian and relevant boundary conditions.** Let $(M, g, x)$ be a pointed compact Riemannian manifold as above, $\dim M = n$. We denote by $L_g = -a_n \Delta_g + R_g$ the conformal Laplacian on $M$, where $a_n = \frac{4(n-1)}{n-2}$.

*The case of closed manifold.* We denote by $\lambda_1(L_g)$ the principal eigenvalue of $L_g$ and denote by $u$ a corresponding positive eigenfunction, normalized as $u(x) = 1$, where $x \in M$ is a base point. It is well-known that the conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has the scalar curvature $R_{\tilde{g}} = u^{-\frac{4}{n-2}} \lambda_1(L_g)$ of the same sign as the principal eigenvalue $\lambda_1(L_g)$. We call $(M, \tilde{g}, x)$ the $L_g$*-conformal satellite of* $(M, g, x)$.

*The case of non-empty boundary.* Here we need relevant boundary conditions. We denote by $h_g$ the *normalized* mean curvature function along the boundary, i.e., $h_g = \frac{1}{n-1} H_g$, where $H_g = \mathrm{tr} A_g$, where $A_g$ is the second fundamental form along $\partial M$. We consider the following pair of operators:

$$\begin{cases} L_g &= -a_n \Delta_g + R_g \quad \text{on } M, \\ B_g &= \partial_\nu + b_n h_g \qquad \text{on } \partial M. \end{cases}$$

where $\partial_\nu$ is the inward normal vector field and $b_n := \frac{n-2}{2}$. Let $s \in [0, 1]$. We consider a Rayleigh quotient and take the infimum:

$$\lambda_1^{(s)} = \inf_{f \in C^\infty_+} \frac{\int_M (a_n |\nabla_g f|^2 + R_g f^2) d\sigma_g + 2(n-1) \int_{\partial M} h_g f^2 d\sigma_g}{(1-s) \cdot \int_{\partial M} f^2 d\sigma_{\partial g} + s \cdot \int_M f^2 d\sigma_g}. \tag{1.1}$$

According to the standard elliptic theory, we obtain an elliptic boundary problem which will be denoted by $(L_{\bar{g}}, B_{\bar{g}})^{(s)}$, and the infimum $\lambda_1^{(s)} = \lambda_1((L_{\bar{g}}, B_{\bar{g}})^{(s)})$ is the *principal eigenvalue of the boundary problem* $(L_{\bar{g}}, B_{\bar{g}})^{(s)}$. We specify the values $s = 0$ and $s = 1$. Then the corresponding Euler-Lagrange equations are the following:

$$
\left\{ \begin{array}{lcll} L_g u & = & 0 & \text{on } M, \\ B_g u & = & \lambda_1^{(0)} u & \text{on } \partial M. \end{array} \right. \qquad \left\{ \begin{array}{lcll} L_g u & = & \lambda_1^{(1)} u & \text{on } M, \\ B_g u & = & 0 & \text{on } \partial M. \end{array} \right. \tag{1.2}
$$

Let $u_0$ and $u_1$ be the corresponding smooth solutions of the systems (1.2) for $s = 0$ and $s = 1$ respectively. It is well-known that the eigenfunctions $u_s$ can be chosen to be positive for all $0 \leq s \leq 1$, see, say, Escobar's work [6, 7, 8]. We always choose the normalization

$$
u_s(x) = 1, \quad s \in \{0, 1\}, \tag{1.3}
$$

for a pointed manifod $(M, g, x)$. Note that this normalization is the same as above in the case of closed manifold.

We consider the conformal metrics $\tilde{g}^{(0)} = u_0^{\frac{4}{n-2}} g$, $\tilde{g}^{(1)} = u_1^{\frac{4}{n-2}} g$ which yield the corresponding scalar and mean curvature functions:

$$
\left\{ \begin{array}{lcll} R_{\tilde{g}^{(0)}} & \equiv & 0 & \text{on } M \\ h_{\tilde{g}^{(0)}} & = & \lambda_1^{(0)} \tilde{u}_0^{-\frac{2}{n-2}} & \text{on } \partial M \end{array} \right. \qquad \left\{ \begin{array}{lcll} R_{\tilde{g}^{(1)}} & = & \lambda_1^{(1)} u_1^{-\frac{4}{n-2}} & \text{on } M \\ h_{\tilde{g}^{(1)}} & \equiv & 0 & \text{on } \partial M. \end{array} \right. \tag{1.4}
$$

We will use the notation:

$$
\mathbb{P}^{(s)} := (L_{\bar{g}}, B_{\bar{g}})^{(s)}, \quad s = 0, 1. \tag{1.5}
$$

**Definition 1.3.** We call the manifold $(M, \tilde{g}^{(0)}, x)$ by *scalar-flat satellite of* $(M, g, x)$, and the manifold $(M, \tilde{g}^{(1)}, x)$ by *minimal boundary satellite of* $(M, g, x)$. In order to have a uniformal terminology, we also call the manifold $(M, \tilde{g}^{(s)}, x)$ by $\mathbb{P}^{(s)}$*-satellite of* $(M, g, x)$, $s = 0, 1$.

1.4. **Satellite sequences.** Here we introduce a concept of *satellite sequences* which plays an important technical role.

**Definition 1.4.** Let $\{(M_i, g_i, x_i)\}$ be a sequence of compact pointed Riemannian manifolds.

(i) If all manifolds $(M_i, g_i, x_i)$ are closed, we denote by $\mathbb{P}$ the conformal Laplacian $L_g$.
(ii) If all manifolds $(M_i, g_i, x_i)$ are with non-empty boundaries, we denote by $\mathbb{P}$ either $\mathbb{P}^{(0)}$ or $\mathbb{P}^{(1)}$ from (1.5).

In all those cases, we write $\mathbb{P}_i := \mathbb{P}_{(M_i, g_i)}$ and denote by $\lambda_1(\mathbb{P}_i)$ the principal eigenvalue and by $u_i^{\mathbb{P}}$ the principal eigenfunction normalized as $u_i^{\mathbb{P}}(x_i) = 1$. Then the sequence $\{(M_i, \tilde{g}_i, x_i)\}$, with $\tilde{g}_i^{\mathbb{P}} := (u_i^{\mathbb{P}})^{\frac{4}{n-2}} g_i$ is called $\mathbb{P}$*-satellite sequence of* $\{(M_i, g_i, x_i)\}$.

1.5. **Main results.** Let $\{(M_i, g_i, x_i)\}$ be a sequence of pointed Riemannian manifolds. We consider two cases: the first case when the manifolds $(M_i, g_i, x_i)$ have empty, and the second one when the manifolds $(M_i, g_i, x_i)$ have non-empty boundaries. In the first theorem, the boundaryless case is already well-known:

**Theorem A.** *Let $\{(M_i, g_i, x_i)\}$ be a sequence of pointed Riemannian manifolds with boundary of dimension $n$ of $(c, k+1)$-bounded geometry (if the boundary is empty, it is enough to assume $(c, k)$-bounded geometry). Then the sequence $\{(M_i, g_i, x_i)\}$ $C^k$-subconverges to a complete pointed manifold $(M_\infty, g_\infty, x_\infty)$ with boundary. Assume furthermore the sequence $\{d_i(x_i, \partial M_i)\}$ is bounded away from zero and infinity. Then $(M_\infty, g_\infty, x_\infty)$ has non-empty boundary.*

In the following Main Theorems, we assume the manifolds $M_i$, whether with or without boundary, to be compact. Here is the result for conformal satellites of closed manifolds:

**Theorem B.** *Let $n \geq 1$, $k \geq 7 + 2n$ and let $\{(M_i, g_i, x_i)\}$ be a sequence of pointed compact Riemannian closed manifolds of dimension $n$ that $C^k$-converges to $(M_\infty, g_\infty, x_\infty)$. Let $\mathbb{P}$ be the conformal Laplacian $L_g$. Let $\mathbb{P}$ be either $\mathbb{P}^{(0)}$ or $\mathbb{P}^{(1)}$ from (1.5), and write $\mathbb{P}_i := \mathbb{P}_{(M_i, g_i)}$. Assume in addition that the sequence of the principal eigenvalues $\{\lambda_1(\mathbb{P}_i)\}$ is bounded. Then*

- (i) *the $\mathbb{P}$-satellite sequence $\{(M_i, \tilde{g}_i^{\mathbb{P}}, x_i)\}$ has $(\tilde{c}, k-5-2n)$-bounded geometry for some $\tilde{c} > 0$;*
- (ii) *the satellite sequence $\{(M_i, \tilde{g}_i^{\mathbb{P}}, x_i)\}$ $C^{k-5-2n}$-subconverges to a complete pointed Riemannian manifold $(\tilde{M}_\infty, \tilde{g}_\infty^{\mathbb{P}}, \tilde{x}_\infty)$;*
- (iii) *there is a diffeomorphism $\phi : (\tilde{M}_\infty, \tilde{x}_\infty) \to (M_\infty, x_\infty)$ such that the metric $\tilde{g}_\infty^{\mathbb{P}}$ is conformal to the metric $\phi^* g_\infty$.*

Now we assume that the manifolds $\{(M_i, g_i, x_i)\}$ have non-empty boundaries. We denote by $d_i$ the distance $d_i(x_i, \partial M_i)$ with respect to the metric $g_i$. Here is our main result for manifolds with non-empty boundaries:

**Theorem C.** *Let $n \geq 1$, $k \geq 8 + 2n$ and let $\{(M_i, g_i, x_i)\}$ be a sequence of pointed compact Riemannian manifolds with non-empty boundaries of dimension $n$ that $C^k$-converges to $(M_\infty, g_\infty, x_\infty)$. Assume the sequence $\{d_i\}$ is bounded away from zero and infinity. Let $\mathbb{P}$ be either $\mathbb{P}^{(0)}$ or $\mathbb{P}^{(1)}$ from (1.5), and write $\mathbb{P}_i := \mathbb{P}_{(M_i, g_i)}$. Assume in addition that the sequence of the principal eigenvalues $\{\lambda_1(\mathbb{P}_i)\}$ is bounded. Then*

- (i) *the $\mathbb{P}$-satellite sequence $\{(M_i, \tilde{g}_i^{\mathbb{P}}, x_i)\}$ has $(\tilde{c}, k-5-2n)$-bounded geometry for some $\tilde{c} > 0$;*
- (ii) *the satellite sequence $\{(M_i, \tilde{g}_i^{\mathbb{P}}, x_i)\}$ $C^{k-6-2n}$-subconverges to a complete pointed Riemannian manifold $(\tilde{M}_\infty, \tilde{g}_\infty^{\mathbb{P}}, \tilde{x}_\infty)$ with non-empty boundary;*
- (iii) *there is a diffeomorphism $\phi : (\tilde{M}_\infty, \tilde{x}_\infty) \to (M_\infty, x_\infty)$ such that the metric $\tilde{g}_\infty^{\mathbb{P}}$ is conformal to the metric $\phi^* g_\infty$.*

**Remark 1.5.** The loss of orders of differentiability in the Main Theorems could certainly be improved, for example by using ring properties of Sobolev spaces instead of Morrey's inequalities, but this is not crucial for our purposes here.

**Remark 1.6.** The additional condition in Theorems B and C have clear geometrical meaning. The first one, that the sequence of principal eigenvalues $\{\lambda_1(\mathbb{P}_i)\}$ is bounded holds for any $(\mathbf{c}, k)$-bounded sequence $\{(M_i, g_i, x_i)\}$ (of manifolds with non-empty boundaries) provided the volume ratio $\frac{\mathrm{Vol}_{g_i} M_i}{\mathrm{Vol}_{\partial g_i} \partial M_i}$ is bounded away from zero and infinity. The second condition, that the distances $d_i$ are bounded, guarantees the limiting manifold $M_\infty$ has non-empty boundary. The sequence $\{(D_i^n, ds^2, 0)\}$ of the Euclidean balls $D_i^n$, $\partial B_i = S^{n-1}$ centered at the origin, of radius $i$, provide an obvious example when the boundary vanishes at the limit. In Theorem C, we singled out the case when the limiting manifolds with bounded geometry have non-empty boundary, and thus the boundary value problems for our elliptic operators make sense.
In Theorem B, when all manifolds $\{(M_i, g_i, x_i)\}$ are closed and compact, the sequence $\{\lambda_1(L_{g_i})\}$ is bounded because of the requirements on bounded geometry of those manifolds.

**Remark 1.7.** The statement of Theorems B, C are not restricted to the conformal Laplacian: From the proof of the main theorem it is clear that the elliptic boundary operator $\mathbb{P}^s$ as above could also be replaced by any linear elliptic differential boundary operator acting on smooth functions on $M$ which depends uniquely, locally and continuously of the metric $g$ and is natural, i.e. covariant with respect to isometries of $g$, if its kernel contains a smooth positive function $u$.

As a corollary to Theorem C, we obtain the following result which goes back to a simple observation that any bounded sequence of numbers subconverges to a non-negative or to a non-positive limit.

**Corollary D.** *Let $\{(M_i, g_i, x_i)\}$ be a pointed $(c, k)$-bounded sequence of Riemannian manifolds with non-empty boundaries of dimension $n$, with $k \geq 8 + 2n$. Assume additionally that all $M_i$ are compact and that the ratio $\frac{\mathrm{Vol}_{g_i} M_i}{\mathrm{Vol}_{\partial g_i} \partial M_i}$ and the distances $d_i = d_i(x_i, \partial M_i)$ are uniformly bounded away from zero and infinity. Then there exists a conformally related sequence $\{(M_i, \tilde{g}_i, x_i)\}$ which $C^{k-6-2n}$-subconverges to a complete Riemannian manifold $(M_\infty, \tilde{g}_\infty, x_\infty)$ with either non-negative or non-positive scalar curvature and minimal boundary.*
*If, in addition, the manifolds $(M_i, g_i, x_i)$ have uniformly bounded diameter or volume, then the limiting manifold $(M_\infty, \tilde{g}_\infty, x_\infty)$ is compact.*

1.6. **Plan of the paper and acknowledgments.** We review necessary results on smooth Cheeger-Gromov convergence in Section 2. Then we prove Theorems B, C and Corollary D in Section 3. In Section 4 (Appendix) we review the flatzoomer technique and prove relevant technical results.
The first named author would like to thank Richard Bamler for illuminating conversations concerning the Cheeger-Gromov convergence. In particular, R. Bamler explained to the first author crucial analytic issues concerning the satellite sequences. Both authors are grateful to Bernd Ammann for insightful discussions and interest in this work.

## 2. Cheeger-Gromov convergence for manifolds with boundary

2.1. **Height functions.** Here we give more details on a convergence for manifolds with boundary. The idea is very simple: for a (in general noncompact) manifold $M$ with boundary, we can always

attach a small collar to get a complete manifold $X$ equipped with a height function $f : X \to (-\infty, 1)$ such that $M = f^{-1}([0,1))$. Then a sequence $\{(M_i, g_i, x_i)\}$ of pointed compact manifolds with non-empty boundary gives a sequence $\{(X_i, g_i, x_i)\}$ (where $g_i$ exends $g_i$ on $M_i$) of complete Riemannian manifolds with additional data: height functions.

**Definition 2.1.** Let $(X, g, x)$ be a pointed Riemannian manifold. A smooth function $f : X \to \mathbb{R}$ is called a $(c,k)$-*height function*, where $k \in \mathbb{N}$, $c > 0$, if the following conditions are satisfied:

(i) $\delta^{\partial}(f) := \min\{\ |\nabla_g f(x)|_g \mid x \in f^{-1}([-\varepsilon, +\varepsilon])\} \geq c^{-1}$, $f^{-1}(\{0\}) \neq \emptyset$, in particular 0 is a regular value for the function $f$;

(ii) $f(x) > 0$, and the distance from the base point $x$ to the submanifold $Y^{(0)} := f^{-1}(0)$ is bounded from below by $c^{-1}$ and by $c$ from above.

(iii) the derivatives $|\nabla^{\ell} f| \leq c$ for all $\ell = 0, 1, \ldots, k$.

A sequence $\{(M_i, g_i, x_i, f_i)\}$ is called of $(c,k)$-bounded geometry if $\{(M_i, g_i, x_i)\}$ is a sequence of $(c,k)$-bounded geometry and $f_i$ are $(c,k)$-height functions on $M_i$.

**Remark 2.2.** Let $(X, g, x)$ and $f : X \to \mathbb{R}$ be as in Definition 2.1. Denote $X^f := f^{-1}([0,1))$. Then by definition, $X^f$ is a smooth manifold with the boundary $\partial X^f = f^{-1}(\{0\}) \neq \emptyset$, i.e., the triple $(X^f, g, x)$ is pointed manifold with non-empty boundary. Here we denote by $g$ the restriction $g|_{X^f}$ to avoid multiple subscripts in sequences. We are interested mostly in the case when the manifold $X^f$ is is compact (at least before taking limits).

**Theorem 2.3.** *Let $\{(X_i, g_i, x_i, f_i)\}$ be a sequence of complete pointed manifolds equipped with height functions of $(c,k)$-bounded geometry with $c > 0$, $k \geq 4$. Then the sequence $\{(X_i, g_i, x_i, f_i)\}$ $C^k$-subconverges to $(X_\infty, g_\infty, x_\infty, f_\infty)$, where $(X_\infty, g_\infty, x_\infty)$ is a complete open manifold, and $f_\infty : X_\infty \to \mathbb{R}$ is a $(c,k)$-height function.*

**Corollary 2.4.** *Let $\{(X_i, g_i, x_i, f_i)\}$ be a sequence from Theorem 2.3. Then, if we denote $M_i := X_i^{f_i}$ for $0 \leq i \leq \infty$, the sequence $\{(M_i, g_i, x_i)\}$ $C^k$-subconverges to a smooth manifold $(M_\infty, g_\infty, x_\infty)$ with non-empty boundary.*

2.2. **Gromov-Hausdorff convergence.** For completeness, we recall some standard definitions following [2, Chapter 3]. Let $Z$ be a metric space, and $Y \subset Z$ be a subspace. Let $B_r(Y)$ be the ball of radius $r$ around $Y$ in $Z$, where $r > 0$. In the case when $Y = \{y\}$, we use the notation $B_r(y)$ instead of $B_r(\{y\})$. Sometimes it will be important to emphasize an ambient space $Z$, then we use the notation $B_r^Z(y)$.

If $Z_0, Z_1 \subset Z$, then the *Hausdorff distance* $d_H(Z_0, Z_1)$ is defined as

$$d_H(Z_0, Z_1) = \inf\{\ r > 0 \mid Z_0 \subset B_r(Z_1), \quad Z_1 \subset B_r(Z_0)\ \},$$

Let $(X, d)$ and $(X', d')$ are metric spaces. Then we say that a continuous map $\phi : X \to X'$ is an $\epsilon$-*isometry* if $||\phi^* d' - d||_\infty < \epsilon$.

**Definition 2.5.** Let $\{(Y_i, d_i, y_i)\}$ be a sequence of pointed proper complete metric spaces. Then the sequence $\{(Y_i, d_i, y_i)\}$ is said to *GH-converges* to a complete and proper metric pointed space $(Y_\infty, d_\infty, y_\infty)$ if one of the following equivalent conditions is satisfied:[2]

(B′) there are sequences $\{r_i\}$, $\{\epsilon_i\}$ of positive real numbers, where $r_i \to \infty$, $\epsilon_i \to 0$, and $\epsilon_i$-isometries $\phi_i : B^{Y_\infty}_{r_i}(y_\infty) \to B^{Y_i}_{r_i}(y_i)$ such that

$$B_{\epsilon_i}(\operatorname{Im}\phi_i) \supset B^{Y_i}_{r_i}(y_i) \quad \text{and} \quad d_i(\phi_i(y_\infty), y_i) < \epsilon_i.$$

(D′) there is a metric space $(Z, d)$ and isometric embeddings $\iota_i : Y_i \to Z$, $\iota_\infty : Y_\infty \to Z$, such that

(i) $\lim_{i\to\infty} \iota_i(y_i) = \iota_\infty(y_\infty)$,

(ii) $\lim_{i\to\infty} d_H(U \cap \iota_i(Y_i), U \cap \iota_\infty(Y_\infty)) = 0$ for any open bounded set $U \subset Z$.

We use the notation $\lim^{GH}_{i\to\infty}(Y_i, d_i, y_i) = (Y_\infty, d_\infty, y_\infty)$.

We need the following fact, which is a particular case of much more general results, see, for example, [2, Proposition 3.1.2, Theorem 3.1.3].

**Theorem 2.6.** *Let $\{(X_i, g_i, x_i)\}$ be a sequence of pointed complete $n$-dimensional Riemannian manifolds such that $\mathrm{Ric}_{g_i} \geq (n-1)\kappa$ for some $\kappa \in \mathbb{R}$ and all $i = 1, 2, \ldots$. Then there exists a pointed proper complete metric space $(Y_\infty, d_\infty, y_\infty)$ such that the sequence $\{(X_i, g_i, x_i)\}$ GH-subconverges to $(Y_\infty, d_\infty, y_\infty)$.*

2.3. **Smooth Cheeger-Gromov convergence.** Let $\{(X_i, g_i, x_i)\}$ be a sequence of pointed complete Riemannain manifolds of dimension $n$ which GH-converges to a metric space $(Y_\infty, d_\infty, y_\infty)$ as in Definition 2.5. Assume that the metric space $(Y_\infty, d_\infty, y_\infty)$ is, in fact, a complete Riemannian manifold, and we use the notation: $(Y_\infty, d_\infty, y_\infty) = (X_\infty, g_\infty, x_\infty)$.

**Definition 2.7.** Assume that a sequence $\{(X_i, g_i, x_i)\}$ GH-converges to a complete Riemannian manifold $(X_\infty, g_\infty, x_\infty)$. Then the sequence $\{(X_i, g_i, x_i)\}$ $C^k$-converges to $(X_\infty, g_\infty, x_\infty)$ if there is an exhaustion of $X_\infty$ by open sets $U_j$, i.e.,

$$U_1 \subset \cdots \subset U_j \subset \cdots \subset X_\infty, \quad X_\infty = \bigcup_j U_j,$$

and there are diffeomorphisms onto their image $\phi_j : U_j \to M_j$ such that $\phi_j \to Id_{X_\infty}$ pointwise, and the metrics

$$\phi_j^* g_j \to g_\infty \quad C^k\text{-converging as } j \to \infty,$$

i.e., there is a point-wise convergence $\phi_j^* g_j \to g_\infty$ and $\nabla^\ell \phi_j^* g_j \to \nabla^\ell g_\infty$ for all $\ell = 1, \ldots, k$, where $\nabla$ denotes the Levi-Civita connection of the metric $g_\infty$ on $X_\infty$.

**Remark 2.8.** Without loss of generalities, we will assume that a system of exhaustions $\{U_j\}$ is nothing but the systems of open balls $\{B_j(x_\infty)\}$ of radius $j = 1, 2, \ldots$, centered at $x_\infty \in X_\infty$.

[2] We skip one more equivalent condition (A′), see [2, Section 3.1.2].

R. Bamler provides a detailed proof (see [2, Theorem 3.2.4]) of the following result:

**Theorem 2.9.** (cf. R. Hamilton [9]) *Let $\{(X_i, g_i, x_i)\}$ be a sequence of pointed complete Riemannian manifolds of dimension $n$. Assume that $\operatorname{inj}_{g_i} \geq c^{-1}$ and $||\nabla^\ell \operatorname{Rm}_{g_i}|| \leq c$ for all $\ell = 0, 1, \dots, k$. Then the sequence $\{(X_i, g_i, x_i)\}$ $C^k$-subconverges to a pointed complete Riemannian manifold $(X_\infty, g_\infty, x_\infty)$ of dimension $n$.*

**Remark 2.10.** Strictly speaking, only the case $k = \infty$ is treated in the Theorems of the references, but their proofs contain implicitly the statement for finite $k$.

2.4. **Proof of Theorem 2.3.** Let $\{(X_i, g_i, x_i, f_i)\}$ be a sequence from Theorem 2.3. By Theorem 2.9 we may assume that the sequence of manifolds $\{(X_i, g_i, x_i)\}$ already $C^k$-converges to a pointed complete Riemannian manifold $(X_\infty, g_\infty, x_\infty)$ of dimension $n$. Without loss of generality, we can also assume that the exhaustions of $X_\infty$ is chosen as a systems of open balls $\{B_i(x_\infty)\}$ of radius $i = 1, 2, \dots$, centered at $x_\infty \in X_\infty$. Let $\phi_i : B_i(x_\infty) \to X_i$ be the diffeomorphisms (on their image) from Definition 2.7.
We recall that $f_i : X_i \to \mathbb{R}$ are $(c, k)$-height functions as in Definition 2.1. By definition, we have that $|\nabla^\ell f_i| \leq c$ for all $\ell = 0, 1, \dots, k$. Then by passing to another subsequence if necessary, the functions $\tilde{f}_i := \phi_i^* f_i$ also $C^k$-converge to a function $f_\infty : X_\infty \to \mathbb{R}$. By assumptions on the sequence of functions $\{f_i\}$, it is evident that the function $f_\infty$ is also a $(c, k)$-height function, and since $\delta^\partial(f_i) \geq c^{-1}$ for all $i = 1, 2, \dots$, we obtain that $\delta^\partial(f_\infty) \geq c^{-1}$. This completes proof of Theorem 2.3.
In particular, we obtain by a quite easy argument that the manifolds $(M_i := X_i^{f_i}, g_i, x_i)$ Hausdorff-converge (in the limit manifold) and thus Gromov-Hausdorff-converge to $(M_\infty := X_\infty^{f_\infty}, g_\infty, x_\infty)$ as pointed metric spaces. However, we need more in order to prove Corollary 2.4: we need to prove Cheeger-Gromov convergence.

**Proposition 2.11.** *Let $(X_i, g_i, x_i)$ be a $C^k$-convergent sequence and let $f_i : X_i \to \mathbb{R}$ be $(c, k)$ height functions, then the sequence $i \mapsto (M_i := X_i^{f_i}, g_{M_i}, x_i)$ of manifolds with boundary $C^{k-1}$-converges to $(M_\infty := X_\infty^{f_\infty}, g_{M_\infty}, x_\infty)$.*

*Proof.* For any fixed radius $r$ and $i > r$, we construct diffeomorphisms $D_i^r$ from the manifold with height function $H := B_r(x_\infty) \cap f_\infty^{-1}(-\varepsilon, \varepsilon)$ to an open set in $B_r(x_\infty) \cap \tilde{f}_i^{-1}(-\varepsilon, \varepsilon)$ (recall that $x_\infty = \Phi_i^{-1}(x_i)$ and that the $\tilde{f}_i$ are defined as in Section 2.4) by means of the gradient flows of $\tilde{f}_i$:

$$D_{i,\partial}^r(x) := \operatorname{Fl}_{\operatorname{grad} \tilde{f}_i}^{t(x)}(x),$$

where $t(x)$ is chosen such that

$$\tilde{f}_i(\operatorname{Fl}_{\operatorname{grad} \tilde{f}_i}^{t(x)}(x)) = f_\infty(x).$$

It is easy to see that $t$ is a smooth function (using the product decomposition of a neighborhood $U$ of $\tilde{f}_i^{-1}(0)$ given by the gradient flow of $\tilde{f}_i$ and the fact that $B_r(x_\infty) \cap f_\infty^{-1}(-\varepsilon, \varepsilon) \subset U$), and $D_{i,\partial}^r$ is

a diffeomorphism on its image. Standard integral estimates yield

$$\varepsilon/2 > |\tilde{f}_i(\mathrm{Fl}^{t(x)}_{\mathrm{grad}\tilde{f}_i}(x)) - \tilde{f}_i(x)| \geq t(x) \cdot \delta^{\partial}(\tilde{f}_i),$$

and with this uniform flow time estimate and the $C^k$-estimates on $\mathrm{grad}\tilde{f}_i$, we get $C^k$-bounds of the $D^r_{i,\partial}$ tending to 0. Now $D^r_{i,\partial}$ is a diffeomorphism from $H$ to its image, which is in $B_{r+1} \cap X^{f_\infty}_\infty$. For $i$ large enough, we get $d(D^r_{i,\partial}y, y)$ smaller than the convexity radius in $B_{r+1}$. This allows us to interpolate $D^r_{i,\partial}$ geodesically with the identity in $\mathrm{int}(M)$: we define

$$D^r_i(y) := \exp_y(\phi(y) \cdot \exp_y^{-1}(D^r_{i,\partial}(y)))$$

for $y \in B_r(x_\infty) \cap \tilde{f}_\infty^{-1}(-\varepsilon, \varepsilon)$ and for a smooth function $\phi$ supported in $B_r(x_\infty) \cap \tilde{f}_\infty^{-1}(-\varepsilon, \varepsilon)$ and identical to 1 in a neighborhood of $\tilde{f}_\infty^{-1}(0)$, and extended by $D^r_i(y) = y$ on the complement, getting a sequence of diffeomorphisms from $M_i \cap B_r$ as above that converges as well. The image of $D^r_i$ is still contained in $B_{r+1} \cap X^{f_\infty}_\infty$ and contains $B_{r-1} \cap X^{f_\infty}_\infty$, which allows to show $M_i \to M_\infty$. □

Now it is easy to see that if $f$ is a $(c, k)$-height function on a manifold of $(c, k)$-bounded geometry, then $X^f = f^{-1}([0, 1))$ is a manifold with boundary of bounded geometry. It is a bit harder to see that actually also the converse is true:

**Theorem 2.12.** *Let $c > 0$ then there exists $\bar{c} > 0$, depending on $c$, such that, for any compact pointed manifold $(M, g, x)$ (with non-empty boundary) of $(c, k)$-bounded geometry, there exists a pointed isometric inclusion $\iota : (M, g, x) \to (X, \bar{g}, x)$ where $(X, \bar{g}, x)$ is a complete open pointed manifold of $(\bar{c}, k)$-bounded geometry and $(\bar{c}, k)$-height function $f$ on $X$ with $\iota(M) = f^{-1}([0, 1))$.*

We postpone a proof of this theorem to Section 3.

## 3. Proofs of Theorems A, B, C, of Corollary D and of Theorem 2.12

3.1. **Rayleigh quotients.** Consider first the case when the manifold $(M, g)$ is compact and closed. Then the principal eigenvalue $\lambda_1(L_g)$ is given by minimizing the Rayleigh quotient

$$\lambda_1(L_g) = \inf_{f \in C^\infty_+(M)} \frac{\int_M (a_n |\nabla f|^2 + R_g f^2) d\sigma_g}{\int_M f^2 d\sigma_g}. \tag{3.1}$$

**Lemma 3.1.** *Let $|R_g|^{\max}$ be the maximum value of $|R_g|$ over $M$. Then*

$$|\lambda_1(L_g)| \leq |R_g|^{\max}. \tag{3.2}$$

*Proof.* Indeed, we use the test function $f = 1$ in (3.1) to see that $\lambda_1(L_g) \leq |R_g|^{\max}$. Then there exists a smooth function $f_0$ such that

$$\lambda_1(L_g) = \frac{\int_M (a_n |\nabla f_0|^2 + R_g f_0^2) d\sigma_g}{\int_M f_0^2 d\sigma_g}.$$

Then we have:

$$\begin{aligned}\lambda_1(L_g) &= \frac{\int_M (a_n|\nabla f_0|^2 + R_g f_0^2)d\sigma_g}{\int_M f_0^2 d\sigma_g} \\ &\geq \frac{-\int_M |R_g| f_0^2 d\sigma_g}{\int_M f_0^2 d\sigma_g} \geq -|R_g|^{\max}.\end{aligned}$$

This proves Lemma 3.1. $\square$

Now we assume that $\partial M \neq \emptyset$, $s \in [0,1]$, then $\lambda_1^{(s)} = \inf\{\ Q^{(s)}(f) \mid f \in C_+^\infty\ \}$, where $Q^{(s)}(f)$ is the Rayleigh quotient:

$$Q^{(s)}(f) := \frac{\int_M (a_n|\nabla_g f|^2 + R_g f^2)d\sigma_g + 2(n-1)\int_{\partial M} h_g f^2 d\sigma_g}{(1-s)\cdot \int_{\partial M} f^2 d\sigma_{\partial g} + s \cdot \int_M f^2 d\sigma_g}. \tag{3.3}$$

The next proposition gives geometric conditions for uniform bounds on $\lambda_1^{(0)}$ and $\lambda_1^{(1)}$.

**Lemma 3.2.** *Let $|R_g|^{\max}$ be the maximum value of $|R_g|$ over $M$ and let $|h_g|^{\max}$ be the maximum value of $|h_g|$ over $\partial M$. Then*

$$|\lambda_1^{(0)}| \leq |R_g|^{\max} \cdot \frac{\mathrm{vol}(M)}{\mathrm{vol}(\partial M)} + 2(n-1)|h_g|^{\max}, \tag{3.4}$$

$$|\lambda_1^{(1)}| \leq |R_g|^{\max} + 2(n-1)|h_g|^{\max} \cdot \frac{\mathrm{vol}(\partial M)}{\mathrm{vol}(M)}. \tag{3.5}$$

*Proof.* We first consider the case $s = 0$. Indeed, we use the test function $f = 1$ in (3.3) to see that

$$\lambda_1(P_g) \leq Q^{(0)}(1) \leq |R_g|^{\max} \cdot \frac{\mathrm{vol}(M)}{\mathrm{vol}(\partial M)} + 2(n-1)|h_g|^{\max}.$$

On the other hand, for a general function $f$, we can estimate $Q^{(0)}(f)$ as

$$Q^{(0)}(f) \geq \frac{-\int_M |R_g| f^2 d\sigma_g + 2(n-1)\int_{\partial M} h_g f^2 d\sigma_g}{\int_{\partial M} f^2 d\sigma_{\partial g}} \geq -|R_g|^{\max}\frac{\mathrm{vol}(M)}{\mathrm{vol}(\partial M)} - 2(n-1)|h_g|^{\max},$$

which yields the claim. The calculation for $s = 1$ is completely analogous. $\square$

3.2. **Tools: Stable versions of classic elliptic estimates.** The aim of this sections is to show that, for a Riemannian manifold-with-boundary $(M, g)$ of dimension $n$ with smooth boundary, the classic elliptic estimates can be made uniform in the metric. We denote the Sobolev and Hölder spaces for a metric $h$ by $H_h^{k,q}$ and $C_h^{s,\alpha}$, respectively. First of all, elementary calculations show that for every Riemannian metric $g$ on $M$, and every $r > 0$, there is a $C > 0$ such that the embeddings $H_g^{k,q} \to H_h^{k,q}$ and $C_g^{s,\alpha} \to C_h^{s,\alpha}$ given by the identity have operator norms bounded by $C$ for all $h \in B_r^{C^k}(g)$. Together with the Morrey estimate at $g$ for $k - nq > s$ (see [1, Theorem 2.30], e.g.) this gives a uniform estimate for $H_h^{k,q} \to H_g^{k,q} \to C_g^{s,\alpha} \to C_h^{s,\alpha}$, thus we have:

**Theorem 3.3. (Stable Morrey's estimate)** *For a Riemannian of dimension $n$ with smooth boundary (possibly, non-empty) and for precompact open sets $U \subset\subset V \subset\subset M$ (which might or might not intersect the boundary), the restriction to $U$ of functions defined almost everywhere in $V$ restricts to a continuous map $H_h^{k,q}(V) \to C_h^{[k-nq]}(U)$ bounded uniformly for $h \in B_r^{C^k(V)}(g)$.* $\square$

Second, we state the following general result, identical to [15, Theorem 5.11.1] except for continuity of $C$. This last part can be seen directly from the proof of Theorem 5.11.1 in [15].

**Theorem 3.4. (Stable Schauder estimates)** *Let $\mathbb{P}$ be an elliptic operator of order $m$ on a Riemannian manifold $(M, g)$ with possibly non-empty boundary, and $u$ be a distribution on $M$, $\mathbb{P}u = f$ with $f \in H^s(M)$. Then $u \in H^{s+m}_{loc}(M)$ and for each $U \subset\subset V \subset\subset M$ and all $\sigma < s + m$, there is an estimate*

$$\|u\|_{H^{s+m}(U)} \leq C\|\mathbb{P}u\|_{H^s(V)} + C\|u\|_{H^\sigma(V)}, \tag{3.6}$$

*where $C$ depends continuously on the $C^1$-norm of the coefficients of $\mathbb{P}$ restricted to $V$.* □

We also get a stable Harnack inequality of the following form:

**Theorem 3.5. (Stable Harnack inequality)** *Let $(M, g)$ be a compact manifold with possibly non-empty boundary. Let $\mathbb{L} := \mathbb{L}^g$ be an elliptic linear operator depending continuously on a metric $g$, where $g$ is a Riemannian metric on $M$. Then for every precompact subset $K \subset M$ there is a constant $A_K > 0$ and a $C^2$-neighborhood $\mathcal{U}$ of $g$ in the space of Riemannian metrics such that for all metrics $h \in \mathcal{U}$ the following inequality holds*

$$\inf\{\ u(x) \mid x \in K\ \} \geq A_K \cdot \sup\{\ u(x) \mid x \in K\ \}. \tag{3.7}$$

*for any positive element $u$ of the kernel of $\mathbb{L} = \mathbb{L}^h$.*

*Proof.* In order to apply Theorem 5.3 in from [12] we use the *geodesic connectedness number* $N(K)$ of $K$, which is defined as the supremum over the number of geodesic balls not intersecting the boundary that one needs to connect a point $p \in K \setminus \partial M$ to another point $q \in K \setminus \partial M$. This number $N(K)$ is finite and $C^1$-stable in the metric, as can be seen by a lower bound of the convexity radius in $K$. Thus we apply [12, Theorem 5.3] to every geodesic ball in such a chain and use the telescope product, which yields $A_K = e^{NC(1+\beta(n+1)^2+K(n+1)^2)}$ in the terminology of [12]. It is easy to see that $C, \beta$ and $K$ depend continuously on the $C^2$-norm of the metric in $K$. □

**Remark 3.6.** We will apply Theorem 3.5 for particular operators $\mathbb{L}$. Namely, in the case of closed manifolds, $\mathbb{L} = L_g - \lambda_1(L_g)$, and in the case of a manifold with non-empty boundary, the operator $\mathbb{L}$ is either $(L_g, B_g - \lambda_1^{(0)})$ or $(L_g - \lambda_1^{(1)}, B_g)$. Then there is a natural choice of function $u$ in the kernel of $\mathbb{L}$, namely a corresponding principal eigenfunction.

3.3. **Extending functions beyond a boundary.** We will need the following technical result allowing us to extend functions beyond the boundary of a manifold in a way that respects infima. To that purpose, let us be given a pointed Riemannian manifold with boundary $(M, g, x)$ of $(c, k)$-bounded geometry.

We would like to construct a standard outer collar to $M$. First, we recall necessary constructions from [13]. Let $(M, g)$ be a Riemannian manifold with non-empty boundary $\partial M$ equipped with the metric $\partial g = g|_{\partial M}$. We denote by $\vec{\nu}$ the inward normal vector field along $\partial M$. Then for a point

$x_0 \in \partial M$ we fix an orthgonormal basis on the tangent space $T_{x_0}\partial M$ to identify it with $\mathbb{R}^{n-1}$. Then for small enough $r_1, r_2 > 0$ there is normal collar coordinates

$$\kappa_{x_0} : B_{r_1}(0) \times [0, r_2) \to M, \quad \kappa_{x_0} : (v, t) \mapsto \exp^g_{\exp^{\partial g}_{x_0}(v)}(t\vec{\nu}), \tag{3.8}$$

where the exponential maps of $\partial M$ and of $M$ are composed. By assumption, the manifold $(M, g)$ has $(c, k)$-bounded geometry, in particular, the boundary $(\partial M, \partial g)$ also has $(c, k)$-bounded geometry. Let us choose a collar $\partial M \times [0, \delta)$ for a small enough $\delta > 0$ such that it is covered by normal collar coordinates charts $U_\ell$ with

$$U_\ell = \kappa_\ell(V_\ell); \qquad V_\ell := W_\ell \times [0, r_2), \qquad W_\ell := B_{r_1^{(\ell)}}(0)$$

where $\kappa_\ell$ is the corresponding map from (3.8). Since the manifold $(M, g)$ has $(c, k)$-bounded geometry, [13, Proposition 3.2] implies that there exist constants $r_0 > 0$ and $c_0$ and positive integer $m_0$ depending only on $c$ and $k$, such that if $r_1, r_2 \le r_0$ the family of charts $\{ \kappa_\ell \mid \ell \in \Lambda \}$ can be chosen locally-finite (this finiteness is controlled by $m_0$), and there is a subordinate partition of unity $\{ \psi_\ell \mid \ell \in \Lambda \}$ satisfying the bound

$$||\psi_\ell||_{C^k} < c_0, \tag{3.9}$$

where, again, $c_0$ only depends on $c$ and $k$. We fix this atlas $\{ U_\ell \mid \ell \in \Lambda \}$ of the interior neck neighborhood once and forever, as well as the subordinate partition of unity $\{ \psi_\ell \mid \ell \in \Lambda \}$, as it can serve as the atlas of a normal neighborhood of metrics close to $g$ as well. Now the atlas $(\kappa_i, \kappa_j^{\text{int}})$ (where the $\kappa_i^{\text{int}}$ are charts for the interior) can be extended to an atlas $(\hat{\kappa}_i, \kappa_j^{\text{int}})$ of a (boundaryless) manifold $X$ diffeomorphic to the interior of $M$ by extending the smooth chart transitions from

$$V_{ij} = W_{ij} \times [0, r_2) \to V_{ji} = W_{ji} \times [0, r_2) \quad \text{to} \quad W_{ji} \times (-\infty, r_2) \to W_{ji} \times (-\infty, r_2)$$

(where $V_{ij} := \kappa_i^{-1}(U_i \cap U_j)$)providing gluing data for a manifold $X$ preserving the bounds (3.9) for the chart transitions. We refer to these atlases as cylindrical atlas and extended cylindrical atlas. Now let $(X, p)$ the extension of the manifold with boundary $M$ as above and let $h$ be a complete Riemannian metric on $X$. Let, furthermore, $0 < r \le \infty$ be given and define $B_r := B_r(x) \subset M$. Let $\Lambda_r \subset \Lambda$ be the subset of boundary chart domains of the cylindrical atlas contained in $B_r$. Then we define $\partial_r M := \bigcup_{\ell \in \Lambda_r} U_\ell$ and let $X_r$ be the union of $M$ and of the images of the extended cylindrical charts belonging to $\Lambda_r$.

**Lemma 3.7. (Stable nonlinear extension operator)** *Let $(X, x)$ the extension of the manifold with boundary $M$ as above and let $h$ be a complete Riemannian metric on $X$, which we can assume to satisfy $\kappa_i^* h > m_0 e$ in every chart. Then there is a map $F : C^0(M, (0, \infty)) \to C^0(X, (0, \infty))$ with the following properties:*

(i) *the map $F$ is an extension operator, i.e., $F(u)|_M = u$ for all $u \in C^0(M, (0, \infty))$, and $F_r(u) := F(u)|_{X_r}$ only depends on $u|_{B_r}$;*

(ii) *for each $k \geq 1$ and each $b > 0$, $F_r$ maps the space $C^k(B_r, (b, \infty))$ to $C^k(X_r, \mathbb{R})$ continuously with respect to the $C^k(B_r)$-norms for an entire open $C^k(B_r)$-neighborhood of metrics;*

(iii) *for each $k \geq 1$, $F_r$ maps $C^k(B_r)$-bounded sets uniformly to $C^k$-bounded sets, i.e., for every $a > 0$ there is a constant $c_1 > 0$ such that*

$$F_r(B_a^{C^k(B_r)}(0) \cap C^k(B_r, (0, \infty))) \subset B_{r \cdot c_1}^{C^k}(0) \subset C^k(X_r)$$

*for a $C^k(B_r)$-neighborhood of metrics.*

*Finally, for every $b > 0$ there is a constant $\beta \in (0, b)$, $\beta = \beta(b)$, such that the bound* $\inf(u|_{B_r}) \geq b$ *implies the bound* $\inf(F_r(u)|_X) \geq \beta$, *uniformly in a $C^k(B_r)$-neighborhood of metrics.*

*Proof.* We use the extension operator $E$ from [14], defined on the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, \infty)$. Namely, let $C^\infty(\mathbb{R}^n_+)$ be the space smooth functions on $\mathbb{R}^n_+$ with uniform convergence on its compact subsets of all derivatives. In [14], Seeley defines a continuous linear extension operator operator $E : C^\infty(\mathbb{R}^n_+) \to C^\infty(\mathbb{R}^n)$. Denote by $\Phi : \mathbb{R}^{n-1} \times [0, r_2) \to \mathbb{R}^{n-1} \times [0, \infty)$, defined by $\Phi(x, r) := (x, \phi(r))$ for $\phi : [0, r_2) \to [0, \infty)$, a stretching diffeomorphism. Define the operator $E_2 := E \circ \Phi : C^\infty(\mathbb{R}^{n-1} \times [0, r_2)) \to C^\infty(\mathbb{R}^n)$. We first extend every member $\psi_\ell$ of the partition of unity by defining

$$\hat{\psi}_\ell \circ \hat{\kappa}_\ell := E_2(\psi_\ell \circ \kappa_\ell),$$

which is well-defined as $E_2$ is a combination of reflections at $\{x_1 = 0\}$, which fits the cylindrical charts, and as $\mathrm{supp}(\psi_\ell) \subset U_\ell$. [3] We define an extension operator $E_M : C^\infty(M) \to C^\infty(X)$ by

$$E_M(u) := \sum \psi_i \cdot (E_2(u \circ \kappa_i) \circ \hat{\kappa}_i^{-1}).$$

It is well-defined for the same reason as above, and by inspection, it is clear from [14] that $E_M$ satisfies the above properties (i), (ii) and (iii) (here we use that the respective metrics in every boundary chart of $M$ satisfy $C^k$-bounds with respect to the Euclidean metric on open subsets of half-spaces.).

However, this construction of the extension map still does not imply a bound $\beta > 0$ on the infimum of $F(u)$ for all $u$ with $\inf(u) > b > 0$. Now, let $u \geq b$ be a function on $B_r$. We define:

$$F(u)(x) := \exp(E_M(\ln u(x))).$$

Indeed, the properties (i)–(iii) can be transferred from $E_M$ to $F$ by using additionally the uniform continuity of $\ln|_{[\sigma,\infty)}$ for any $\sigma > 0$. Since $\ln|_{[\sigma,\infty)}$ is bounded away from $-\infty$, there exists $\beta = \beta(b)$ such that $\inf(F(u)|_{X_r}) \geq \beta$ provided $\inf(u|_{B_r}) \geq b$. $\square$

[3] The family of the $\hat{\psi}_\ell$ is not a partition of unity beyond the boundary any more but still its sum nowhere vanishes and the family is locally finite, thus by the usual normalization procedure the family could be made a partition of unity. However, we do not need this property here.

3.4. **Proof of Theorem B.** Let $\{(M_i, g_i, x_i)\}$ be a sequence of compact closed manifolds. We denote $\mathbb{P}_i = L_{g_i}$, and let $\lambda_1(\mathbb{P}_i)$ be the principal eigenvalue, and $u_i$ be a corresponding eigenfunction normalized as $u_i(x_i) = 1$. Since the sequence $\{(M_i, g_i, x_i)\}$ has $(c, k)$-bounded geometry, then by Lemma 3.1, the sequence $\{\lambda_1(\mathbb{P}_i)\}$ is also bounded by $a(n) \cdot c$, where $a(n)$ depends only on $n$. Let $(M_\infty, g_\infty, x_\infty)$ be the limit of the sequence $\{(M_i, g_i, x_i)\}$ with diffeomorphisms $\phi_i^{(m)} : B_m(x_\infty) \to B_m(x_i)$, where $B_m(x_i) \subset M_i$. We denote by $\tilde{B}_m(x_i) \subset M_i$ some subset with smooth boundary such that

$$B_m(x_i) \subset \tilde{B}_m(x_i) \subset B_{m+1}(x_i).$$

Let $u_i^{(m)}$ be the restriction of the function $u_i$ to $B_m(x_i)$. We set $K := B_{m+1}(x_i)$ and $M$ to be the closure of $\tilde{B}_{m+2}(x_i)$ and apply the stable Harnack inequality from Theorem 3.5 for the pair $K \subset M$. Therefore, as the normalization of the functions $u_i^{(m)}$ implies

$$\sup\{\ u_i^{(m)}(x) \mid x \in B_{m+1}\ \} \geq 1 \geq \inf\{\ u_i^{(m)}(x) \mid x \in B_{m+1}\ \},$$

we obtain uniform point-wise bounds of the $(\phi_i^{(m)})^*(u_i^{(m)})$ away from zero and infinity on the ball $B_{m+1}(x_\infty)$. The latter imply uniform $L^2$-bounds depending on $m$.

Now we apply the stable Schauder estimate from Theorem 3.4 to the case of $\mathbb{P}_i := (\phi_i^{(m)})^* L_{g_i}$, $U := B_m(x_\infty)$, $V := B_{m+1}(x_\infty)$, showing that there is a single constant $C$ for all of the involved Schauder estimates. Together with the established $L^2$-bounds we obtain that there are constants $C_l^{(m)} > 0$ such that

$$||u_i||_{H^l(B_{m+1}(x_i))} \leq C_l^{(m)} \tag{3.10}$$

for all $i \in \mathbb{N}$ up to $l = k-3$, as the $C^k$-norm of the coefficients of the conformal Laplacian depends on the $C^{k+2}$-norm of the metric, and the Schauder estimates require a $C^1$ control over the coefficients. Then the stable Morrey's estimate from Theorem 3.3 and the elliptic estimates Eq. 3.10 above imply that

$$|u_i|_{C^l(B_{m+1}(x_i))} \leq \tilde{C}_l^{(m)}. \tag{3.11}$$

for some constants $\tilde{C}_l^{(m)} > 0$ for all $i \in \mathbb{N}$ and all $l \leq k - 3 - 2n$. Furthermore, Theorem 3.5 implies that we also have a uniform estimate of $u_i$ from 0.

Let $\tilde{g}_i := (u_i)^{\frac{4}{n-2}} \cdot g_i$, then Theorem 4.2 and Theorem 4.11 say[4] that $\mathrm{inj}^{-1}$ is a uniform quasi-flatzoomer of degree 2 and $|\nabla^l R^g|_g$ is a uniform quasi-flatzoomer of degree $2 + l$. Consequently,

$$|\operatorname{Rm}_{\tilde{g}_i^{(m)}}|_{C^l} \quad \text{and} \quad \mathrm{inj}^{-1}_{\tilde{g}_i^{(m)}}$$

are uniformly bounded for all $l \leq k - 5 - 2n$. Then by Theorem 2.9 there is a subsequence $\{(M_i, g_i, x_i)\}$ such that the restrictions of the metrics $\tilde{g}_i := (u_i)^{\frac{4}{n-2}} \cdot g_i$ to $B_m(x_i)$ converge in $C^{k-5-2n}$.

[4] Note that for the sake of greater consistency with the article [11], in the Appendix we maintain the convention that the conformal factor is $e^{2u}$ instead of a power of $u$ as before. Therefore, the bounds from zero and infinity for $u$ here imply bounds from $\pm\infty$ in the Appendix, which is needed to bound the value of the quasi-flatzoomer.

Taking inductively subsequences for every $m \in \mathbb{N}$, the diagonal sequence is finally a subsequence that converges in Hamilton's sense. It is easily seen that the metric $\tilde{g}_\infty$ is conformal to $g_\infty$, as the conformal distortion between the metrics $(\phi_i^{(m)})^* g_i$ and $\tilde{g}_\infty$ tends to zero with $i \to \infty$ for every $m$. Here the conformal distortion $\mathsf{conf\text{-}distortion}_x(g_1, g_2)$ between two Riemannian metrics $g_1, g_2$ at a point $x$ is defined by

$$\mathsf{conf\text{-}distortion}_x(g_1, g_2) := \sup\left\{ \tfrac{g_2(v,v)}{g_1(v,v)} - \tfrac{g_2(w,w)}{g_1(w,w)} \;\middle|\; v, w \in T_pM \setminus \{0\} \right\}.$$

This proves Theorem B. □

3.5. **Proof of Theorem C.** For every $i$, we solve the principal-value boundary problem given by the operator $\mathbb{P}_i$ on $M_i$ to find its principal eigenvalue $\lambda_1(\mathbb{P}_i)$ and a smooth positive eigenfunction $u_i$ normalized by $u_i(x_i) = 1$. Then we construct the conformal metric $\tilde{g}_i = u_i^{\frac{4}{n-2}} g_i$, and we obtain the satellite sequence $\{(M_i, \tilde{g}_i, x_i)\}$. Then, as in the proof of Theorem A, the estimates in Theorems 3.3, 3.4 and Theorem 3.5 as well as Theorem 4.2 and Theorem 4.11 imply that there exists $c'$ such that all manifolds $(M_i, \tilde{g}_i, x_i)$ have $(c', k-5-2n)$-bounded geometry.

Now again use Theorem 2.12 to construct a $(c', k-5-2n)$-bounded sequence of complete non-compact manifolds $\{(\tilde{X}_i, \tilde{g}_i, x_i)\}$ with height functions $f_i$, such that $(M_i, g_i, x_i) \subset (\tilde{X}_i, \tilde{g}_i, x_i)$, where the metric $\tilde{g}_i$ on $X_i$ extends $g_i$ on $M_i$. As above, the $(c', k-5-2n)$-height-functions $\tilde{f}_i$ on $\tilde{X}_i$ are such that $M_i := X_i^{f_i} := \tilde{f}_i^{-1}([0,1))$. We fix the embedding $\tilde{\iota}_i : M_i \to \tilde{X}_i$, such that

(1) $\tilde{\iota}_i(x_i) = x_i$,
(2) $\tilde{\iota}_i(M_i) = \tilde{f}_i^{-1}([0,1))$,
(3) $\tilde{\iota}_i^* \tilde{g}_i = \tilde{g}_i = u_i^{\frac{4}{n-2}} g_i$.

Now, using Theorem 2.3, we pass to a subsequence, called $\{(\tilde{X}_i, \tilde{g}_i, x_i)\}$ again, that $C^{k-6-2n}$-converges to the manifold $\{(\tilde{X}_\infty, \tilde{g}_\infty, x_\infty)\}$ and such that the sequences of the height-functions $\{\tilde{f}_i\}$ converge as well. In particular, we obtain the following commutative diagram of convergent sequences:

$$\begin{tikzcd}
(\tilde{X}_i, \tilde{g}_i, x_i) \arrow[r, "i\to\infty"] & (\tilde{X}_\infty, g_\infty, x_\infty) \\
(\tilde{f}_i^{-1}([0,1)), \tilde{g}_i, x_i) \arrow[u, "\tilde{\iota}_i"] \arrow[r, "i\to\infty"] & (\tilde{f}_\infty^{-1}([0,1)), g_\infty, x_\infty) \arrow[u, "\tilde{\iota}_\infty"'] \\
(M_i, \tilde{g}_i, x_i) \arrow[u, leftrightarrow, "Id_i"] \arrow[r, "i\to\infty"] & (M_i, g_\infty, x_\infty) \arrow[u, leftrightarrow, "\tilde{I}d_\infty"']
\end{tikzcd}$$

Here $Id_i^* \tilde{g}_i = \tilde{g}_i = u_i^{\frac{4}{n-2}} g_i$. We choose yet one more subsequence such that the functions $u_i$ are $C^{k-6-2n}$-converging to a smooth function $u_\infty$ on $M_\infty = \tilde{f}_\infty^{-1}([0,1))$. This, together with the Theorem 2.11 and the consideration of the conformal distortion as before, proves Theorem C. □

3.6. **Proof of Theorem D.** By assumptions, the sequence $\{(M_i, g_i, x_i)\}$ has $(\mathbf{c}, k)$-bounded geometry. Consider the operators $\mathbb{P}_i^{(1)} = (L_{g_i}, B_{g_i})^{(1)}$. Then Lemma 3.2 implies that the sequence of

eigenvalues $\{\lambda_1(\mathbb{P}_i^{(1)}\}$ is bounded. By passing to a subsequence, we can assume that $\lambda_1(\mathbb{P}_i^{(1)})$ converge to $\lambda_1$. Assume that $\lambda_1 \geq 0$. Then Theorem B implies that the conformal satellites $(M_i, \tilde{g}_i^{\mathbb{P}}, x_i)$ converge to a manifold $(M_\infty, g_\infty, x_\infty)$ with $R_{g_\infty} \geq 0$. If $\lambda_1 \leq 0$ we get a similar conclusion. □

3.7. **Proof of Theorem 2.12.** This is basically an extension of the proof of Lemma 3.7 to endomorphism-valued functions. Let $\kappa_i : V_i \to U_i$ be a member of the cylindrical atlas. We denote by $g_\ell$ the metric $g$ restricted to $U_\ell$, and by $e_\ell = (\kappa_{(\ell)}^{-1})^*(ds^2)$, where $ds^2$ is the Euclidean metric on $B_{r_1^{(\ell)}}(0) \times [0, r_2))$. For each $\ell$, we define the operator

$$A_\ell := e_\ell^{-1} \circ g_\ell : TU_\ell \to TU_\ell \tag{3.12}$$

where the metrics are understood as maps $TU_\ell \to T^*U_\ell$. The operators $A_\ell$ are positive-definite symmetric operators, their spectrum is therefore contained in $(0, \infty)$. In [13, Proposition 2.3], it is shown that there is constant $a_0 > 0$ such that the norms $|A_\ell|$ are uniformly bounded by $a_0$ away from $\infty$ and by $a_0^{-1} > 0$ from 0. This allows to define the maps $\mathbf{a}_\ell := \ln(A_\ell)$, which are smooth maps from $U_i$ to the set of symmetric matrices $\mathrm{Mat}_s(\mathbb{R}^n, \mathbb{R}^n)$ bounded by $\ln a_0$. We use the Seeley operator $F$ from Lemma 3.7 to extend the coefficients of each matrix $\mathbf{a}_\ell$ to the members of the extended atlas

$$\hat{U}_\ell := \kappa_\ell(V_\ell), \qquad V_\ell := B_{r_1^{(\ell)}}(0) \times (-\infty, r_2)).$$

This gives maps $\hat{\mathbf{a}}_\ell : \hat{U}_\ell \to \mathrm{Mat}_s(\mathbb{R}^n, \mathbb{R}^n)$, such that $\hat{\mathbf{a}}_\ell|_{U_\ell} = \mathbf{a}_\ell$.
Equally we define $\hat{\psi}_\ell$ as the Seeley extensions of the partition of unity $\psi_i$. Then we define $\hat{A}_\ell := \exp(\hat{\mathbf{a}}_\ell)$, which is a positive-definite symmetric smooth extension of $A_\ell$. Thus we can define the Riemannian metric $\hat{g}_\ell := e_\ell \circ \hat{A}_\ell$. Finally, we put $\hat{g} := \sum \hat{\psi}_\ell \cdot \hat{g}_\ell$, which is complete metric on the open manifold $X$. Now let $e_\ell$ be the Euclidean metric in the new chart $\hat{U}_\ell$, and define the metric $\bar{g}_\ell := \hat{\kappa}_\ell^* \hat{g}$ on

$$B_{r_1^{(\ell)}}(0) \times (-\infty, r_2)) \subset \mathbb{R}^n.$$

Denote $\bar{A}_\ell := \bar{g}_\ell$. By construction, each operator $\bar{A}_\ell$ is has norm bounded away from infinity. But also the norm of its inverse is bounded: Since each point $x \in \hat{M}$ there at most $m_0$ neighborhoods $U_\ell$ such that $x \in U_\ell$. Consequently, there is an index $\ell_0$ such that $\psi_{\ell_0}(x) \geq m_0^{-1}$ and thus we have

$$e(\bar{A}_\ell v, v) = \sum_{\ell'} \hat{\psi}_{\ell'} \hat{A}_{\ell'}(v, v) \geq m_0^{-1} \hat{A}_{\ell_0}(v, v) \tag{3.13}$$

for some $\ell_0$, as all summands are positive. Now $\hat{A}_{\ell_0}(v, v)$ in turn can be estimated by

$$\tilde{A}_{\ell_0}(v, v) \geq ||\tilde{B}_{\ell_0}|| \cdot ||v||. \tag{3.14}$$

These are exactly the estimates needed to show bounded geometry of $(X, \hat{g})$. As a height function we take, for $\tau \in C^\infty(-\infty, r_2])$ with $\tau(r) = r$ for all $r \in (-\infty, r_2/4]$ and $\tau([r_2/2, r_2]) = r_2/2$,

$$f := \sum_\ell \psi_\ell \cdot (\tau \circ x_1 \circ \kappa_\ell^{-1}), \tag{3.15}$$

complemented by $r_2/2$ in the interior, which is easily seen to satisfy all our requirements. □

3.8. **Proof of Theorem A.** Now let us prove Theorem A. Assume that we are given a sequence of pointed manifold with boundary $(M_i, g_i, x_i)$ of $(\mathbf{c}, k)$-bounded geometry. Then we can extend every $(M_i, g_i, x_i)$ to a pointed boundaryless manifold $(X_i, \overline{g}_i, x_i, f_i)$ as in Theorem 2.12. Then Theorem 2.9 implies that there is a convergent subsequence for both manifolds and height functions, also denoted by $(X_i, \overline{g}_i, x_i, f_i)$. Finally, Proposition 2.11 implies that, in the $C^k$ sense,

$$\lim_{i\to\infty}(M_i, g_i, x_i) = \lim_{i\to\infty}(X_i^{f_i}, \overline{g}_i, x_i) = (X_\infty^{f_\infty}, \overline{g}_\infty, x_i),$$

which proves Theorem A. □

## 4. Appendix: Uniform flatzoomers

In this section, we define the notions *uniform flatzoomer* and *uniform quasi-flatzoomer*. This is a direct adaptation and sharpening of the results in [11], with only slightly modified proofs. The crucial difference is that in the following we also have to show local uniformity in the metric (which for simplicity is assumed to be Riemannian, in contrast to the very general setting in [11]).

4.1. **Notations and definition.** Let $(M, g)$ be a Riemannian manifold. We denote by $\mathcal{R}\mathrm{iem}(M)$ the space of all Riemannian metrics. We introduce the following notations:

- For $u \in C^\infty(M, \mathbb{R})$, we denote the Riemannian metric $\mathrm{e}^{2u}g$ by $g[u]$.
- For $i \in \mathbb{N}$, the $i$th covariant derivative with respect to $g$ of a $C^\infty$ tensor field $T$ on $M$ is denoted by $\nabla_g^i T$.
- The function $\langle T, T\rangle_g \in C^\infty(M, \mathbb{R})$ is the total contraction of $T \otimes T$ via $g$ in corresponding tensor indices. If $T$ is for instance a field of $k$-multilinear forms, this means that for every $x \in M$ and every $g$-orthonormal basis $(e_1, \ldots, e_n)$ of $T_xM$, we have

$$\langle T, T\rangle_g(x) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} T(e_{i_1}, \ldots, e_{i_k})^2 .$$

- The function $|T|_g \in C^0(M, \mathbb{R}_{\geq 0})$ is defined to be $\left|\langle T, T\rangle_g\right|^{1/2}$.
- The bundle $s : S \to M$ is defined as $S := \tau_M^* \tilde{\otimes} \tau_M^*$ where $\tilde{\otimes}$ is the symmetric tensor product.
- For any vector bundle $\pi : E \to B$ over a manifold $B$, we denote its $l$-th jet bundle by $J^l(E)$.
- $\mathrm{Rm}_g$ denotes the Riemann tensor, viewed as a tensor field of type $(4, 0)$. We adopt the Besse sign convention for $\mathrm{Rm}_g$ [3].
- For $m, d \in \mathbb{N}$, $\mathcal{P}_m^d$ denotes the (finite-dimensional) vector space of real polynomials of degree $\leq d$ in $m$ variables, equipped with its unique Hilbert space topology.

**Definition 4.1.** Let $M$ be a manifold. A functional $\mathbf{\Phi}\colon C^\infty(M, \mathbb{R}) \times \mathcal{R}\mathrm{iem}(M) \to C^0(M, \mathbb{R}_{\geq 0})$ is a *uniform Riemannian flatzoomer of degree* $k$ if and only if for some metric $\eta$ on $M$, there exist

$k, l, d \in \mathbb{N}$, a continuous positive function $\alpha : J^l(S) \to \mathbb{R}_{>0}$, $u_0 \in C^0(M, \mathbb{R})$ and a continuous map $J^l S) \to \mathcal{P}^d_{k+1}$ such that

$$\mathbf{\Phi}(u, g)(x) \;\leq\; \mathrm{e}^{-\alpha(g)\cdot u(x)} P_g(x) \left(u(x),\, |\nabla^1_\eta u|_\eta(x),\, \ldots,\, |\nabla^k_\eta u|_\eta(x)\right)$$

holds for all $x \in M$ and all $u \in C^\infty(M, \mathbb{R})$ which satisfy $u(x) > u_0(x)$.

It is easy to prove that the notion of uniform Riemannian flatzoomer does not depend on the choice of the metric $\eta$.

4.2. **Norm of the Riemannian tensor as a uniform flatzoomer.** Below we adopt the following convention: if we fix a metric $g$, then a flatzoomer is written as $\mathbf{\Phi}\colon C^\infty(M, \mathbb{R}) \to C^0(M, \mathbb{R}_{\geq 0})$.

**Theorem 4.2.** *Let $(M, g)$ be a Riemannian manifold and $k \in \mathbb{N}$. Then $\mathbf{\Phi}\colon C^\infty(M, \mathbb{R}) \to C^0(M, \mathbb{R}_{\geq 0})$ defined by*

$$\mathbf{\Phi}(u) := \left|\nabla^k_{g[u]} \mathrm{Rm}_{g[u]}\right|_{g[u]}$$

*is a uniform flatzoomer of degree $k + 2$.*

*Proof of Lemma 4.2.* For a positive integer $r$, let $\mathcal{T}_r(M) \to M$ denote the real vector bundle of $(r, 0)$-tensors on $M$.
For $u \in C^\infty(M, \mathbb{R})$, the $(4, 0)$-Riemann curvature of the conformal metric $g[u]$ is given as

$$\mathrm{Rm}_{g[u]} = \mathrm{e}^{2u} \left(\mathrm{Rm}_g - g \owedge \left(\mathrm{Hess}_g\, u - \mathrm{d}u \otimes \mathrm{d}u + \tfrac{1}{2}|\mathrm{d}u|^2_g\, g\right)\right) , \tag{4.1}$$

see [3, Theorem 1.159b]. Let $v$ be a vector field. We have:

$$\nabla^{g[u]}_v X = \nabla^g_v X + \mathrm{d}u(X)v + \mathrm{d}u(v)X - g(v, X)\, \mathrm{grad}_g\, u\,, \tag{4.2}$$

see [3, Theorem 1.159a].
Let $k, m$ be nonnegative integers. We consider two types of basic morphisms of tensor bundles

$$\pi\colon \mathcal{T}_{k+4+2m}(M) \to \mathcal{T}_{k+4+2m}(M) \quad \text{and} \quad \xi\colon \mathcal{T}_{k+4+2m}(M) \to \mathcal{T}_{k+4}(M). \tag{4.3}$$

Here $\pi$ is given by a permutation (the same over each $x \in M$) of the tensor indices, and $\xi$ contracts each of the first $m$ pairs of indices via the metric $g$. Clearly the morphisms $\pi$ and $\xi$ and their compositions depend continuously on the 0-jet of the metric $g$.

**Definition 4.3.** Let $k, m$ be nonnegative integers, and $\mathrm{PC}^g_{k,m}$ be a real vector space spanned by all morphisms of the form $\xi \circ \pi$, where $\pi$ and $\xi$ are as in (4.3).

By construction, the space $\mathrm{PC}^g_{k,m}$ is finitely-dimensional. In [11], Claim in proof of 2.5., the authors prove:

**Proposition 4.4.** *For every $k \in \mathbb{N}$, there exists a number $\mu_k \geq 1$ and, for every $i \in \{1, \ldots, \mu_k\}$,*

- *there exist an integer $a_{k,i} \geq 1$ and a section $\omega_{k,i}$ of the bundle $\mathcal{T}_{a_{k,i}}(M) \to M$,*
- *there exist integers $c_{k,i,1}, \ldots, c_{k,i,k+2} \geq 1$,*

- *there exists a number* $m_{k,i} \geq 1$ *with* $a_{k,i} + \sum_{\nu=1}^{k+2} \nu c_{k,i,\nu} = k + 4 + 2m_{k,i}$,
- *and there exists a morphism* $\psi_{k,i} \in \mathrm{PC}^g_{k,m_{k,i}}$

*such that the following equation holds for all* $u \in C^\infty(M, \mathbb{R})$*:*

$$\nabla^k_{g[u]} \mathrm{Rm}_{g[u]} \ = \ \mathrm{e}^{2u} \sum_{i=1}^{\mu_k} \psi_{k,i} \left( \omega_{k,i} \otimes (\nabla^1_g u)^{\otimes c_{k,i,1}} \otimes \cdots \otimes (\nabla^{k+2}_g u)^{\otimes c_{k,i,k+2}} \right) . \tag{4.4}$$

Moreover, inspection of the proof in [11] yields immediately that the first three items above are independent of the metric used, whereas $\psi_{k,i}$, being an element of $\mathrm{PC}^g_{k,m}$, depends continuously on the metric, as mentioned above.

Now let $u \in C^\infty(M, \mathbb{R})$. To compute $\mathbf{\Phi}(u)$ at a point $x \in M$, we choose an $h$-orthonormal basis $(e_1, \ldots, e_n)$ of $T_xM$. Then $(e_1[u], \ldots, e_n[u])$ defined by $e_i[u] := \mathrm{e}^{-u} e_i$ is an $h[u]$-orthonormal basis of $T_xM$. Then

$$\begin{aligned}
\mathbf{\Phi}(u) &= \left| \nabla^k_{g[u]} \mathrm{Rm}_{g[u]} \right|_{h[u]} = \left| \sum_{a \in \{1,\ldots,n\}^{k+4}} (\nabla^k_{g[u]} \mathrm{Rm}_{g[u]}) \left( e_{a_1}[u], \ldots, e_{a_{k+4}}[u] \right)^2 \right|^{1/2} \\
&= \left| \mathrm{e}^{-2(k+4)u} \sum_{a \in \{1,\ldots,n\}^{k+4}} (\nabla^k_{g[u]} \mathrm{Rm}_{g[u]}) \left( e_{a_1}, \ldots, e_{a_{k+4}} \right)^2 \right|^{1/2} = \mathrm{e}^{-(k+4)u} \left| \nabla^k_{g[u]} \mathrm{Rm}_{g[u]} \right|_h .
\end{aligned}$$

Let $\eta$ be any Riemannian metric on $M$. Now we use (4.4) to conclude that there exists a polynomial $P \in C^0(M, \mathcal{P}^d_{k+2})$ of suitable degree $d$, such that

$$\begin{aligned}
\mathbf{\Phi}(u)(x) &= \mathrm{e}^{-(k+4)u(x)} \left| \nabla^k_{g[u]} \mathrm{Rm}_{g[u]} \right|_h (x) \\
&= \mathrm{e}^{-(k+2)u(x)} \left| \sum_{i=1}^{\mu_k} \psi_{k,i}(\omega_{k,i} \otimes (\nabla^1_g u)^{\otimes c_{k,i,1}} \otimes \cdots \otimes (\nabla^{k+2}_g u)^{\otimes c_{k,i,k+2}}) \right|_h (x) \\
&\leq \mathrm{e}^{-(k+2)u(x)} P(x)(|\nabla^1_\eta u|_\eta(x), \ldots, |\nabla^{k+2}_\eta u|_\eta(x)) .
\end{aligned}$$

Furthermore, the polynomial $P$ does not depend on $u$ but does depend continuously and pointwise on the metric $\eta$. Hence $\mathbf{\Phi}$ is a uniform Riemannian flatzoomer. $\square$

4.3. **Composition of uniform flatzoomers.** Let $M$ be a manifold, let $m \in \mathbb{N}$. For $i \in \{1, \ldots, m\}$, let $\mathbf{\Phi}_i \colon C^\infty(M, \mathbb{R}) \to C^0(M, \mathbb{R}_{\geq 0})$ be a flatzoomer.

**Definition 4.5.** We say that a function $Q \in C^0(M \times (\mathbb{R}_{\geq 0})^m, \mathbb{R}_{\geq 0})$ is *homogeneous-polynomially bounded* if there exist $r \in \mathbb{R}_{>0}$ and $c \in C^0(M, \mathbb{R}_{\geq 0})$ with

$$\forall x \in M \colon \forall v_1, \ldots, v_m \in [0,1] \colon Q(x, v_1, \ldots, v_m) \leq c(x) \cdot (v_1 + \cdots + v_m)^r .$$

Now we use $Q$ to compose the flatzoomers $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_m$ by defining a new functional $\mathbf{\Phi} \colon C^\infty(M, \mathbb{R}) \to C^0(M, \mathbb{R}_{\geq 0})$ as follows:

$$\mathbf{\Phi}(u)(x) := Q\left(x, \mathbf{\Phi}_1(u)(x), \ldots, \mathbf{\Phi}_m(u)(x)\right) . \tag{4.5}$$

**Lemma 4.6.** *Assume the flatzoomers $\mathbf{\Phi}_i\colon C^\infty(M,\mathbb{R}) \to C^0(M,\mathbb{R}_{\geq 0})$ as above are uniform of degree $k_i$. Then the functional $\mathbf{\Phi}\colon C^\infty(M,\mathbb{R}) \to C^0(M,\mathbb{R}_{\geq 0})$ defined by (4.5) is a uniform flatzoomer of degree $\leq \max\{k_1,...,k_m\}$.*

This applies in particular to the function $Q$ given by $Q(x,v) = \sum_{i=1}^m v_i$. Thus $\mathbf{\Phi} := \sum_{i=1}^m \mathbf{\Phi}_i$ is a uniform flatzoomer. In this way, finitely many uniform flatzoomers can be controlled by a single uniform flatzoomer: if $\mathbf{\Phi}(u) \leq \varepsilon$ holds for some $u \in C^\infty(M,\mathbb{R})$ and $\varepsilon \in C^0(M,\mathbb{R}_{>0})$, then $\mathbf{\Phi}_i(u) \leq \varepsilon$ for every $i \in \{1,\dots,m\}$. Another example is obtained by taking $m=1$ and $Q(x,s) = s^{1/2}$.

*Proof.* Let $\eta$ be a Riemannian metric on $M$. For each $i \in \{1,\dots,m\}$, there exist $k_i, d_i \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_{>0}$ and $b_i, u_i \in C^0(M,\mathbb{R}_{\geq 0})$ such that

$$\mathbf{\Phi}_i(u)(x) \;\leq\; \mathrm{e}^{-\alpha_i u(x)}\, b_i(x) \cdot \left(1 + \sum_{j=0}^{k_i} |\nabla_\eta^j u|_\eta(x)\right)^{d_i}$$

holds for all $l \in \mathbb{N}$ and $x$ and $u \in C^\infty(M,\mathbb{R})$ which satisfy $u > u_i$. We consider $k := \max\{k_1,\dots,k_m\}$, $d := \max\{d_1,\dots,d_m\}$, $\alpha := \min\{\alpha_1,\dots,\alpha_m\}$ and the pointwise maxima $u_0 := \max\{u_1,\dots,u_m\}$, $b := \max\{b_1,\dots,b_m\}$ in $C^0(M,\mathbb{R}_{\geq 0})$. For every $i \in \{1,\dots,m\}$,

$$\mathbf{\Phi}_i(u)(x) \;\leq\; \mathrm{e}^{-\alpha u(x)}\, b(x) \cdot \left(1 + \sum_{j=0}^{k} |\nabla_\eta^j u|_\eta(x)\right)^{d}$$

holds for all $l \in \mathbb{N}$ and $x$ and $u \in C^\infty(M,\mathbb{R})$ which satisfy $u > u_0$. This implies for all $l \in \mathbb{N}$ and $x$ and $u > u_0$:

$$\begin{aligned}
\mathbf{\Phi}(u)(x) \;&=\; Q\left(x, \mathbf{\Phi}_1(u)(x), \dots, \mathbf{\Phi}_m(u)(x)\right) \;\leq\; c(x) \cdot \left(\mathbf{\Phi}_1(u)(x) + \cdots + \mathbf{\Phi}_m(u)(x)\right)^r \\
&\leq\; m^r c(x) \left( b(x)\, \mathrm{e}^{-\alpha u(x)} \left(1 + \sum_{j=0}^{k} |\nabla_\eta^j u|_\eta(x)\right)^{d} \right)^r \\
&\leq\; \mathrm{e}^{-\alpha r u(x)}\, m^r c(x) b(x)^r \left(1 + \sum_{j=0}^{k} |\nabla_\eta^j u|_\eta(y)\right)^{dr}.
\end{aligned}$$

Thus $\mathbf{\Phi}$ is a uniform flatzoomer. ☐

4.4. **Uniform quasiflatzoomers and injectivity radius.** Here our goal is to provide uniform estimates for the injectivity radius of the conformal metrics.

**Definition 4.7.** Let $M$ be a manifold. A functional $\boldsymbol{\phi}\colon C^\infty(M,\mathbb{R}) \times \mathcal{R}\mathrm{iem}(M) \to [0,\infty]$ is a *uniform quasiflatzoomer of degree* $k$ if for some Riemannian metric $\eta$ on $M$ and for any $c > 0$, there exist $k \in \mathbb{N}$ and a continuous function $F : [0,\infty) \to [0,\infty)$ such that for every compact subset $L \subset M$ there is a compact subset $K \subset M$ such that for every metric $h \in \mathcal{R}\mathrm{iem}(M)$ there exists a $C^{k-1}(K)$-neighborhood $U(h)$ such that the inequality

$$\sup\{\boldsymbol{\phi}(u,g)(x) | x \in M\} \leq F(||\nabla u||_{C^{k-1}(K,\eta)})$$

holds for all $g \in U(h)$ and $u \in C^\infty(M,\mathbb{R})$ with $u > c$.

Again, it easy to show that a functional $\boldsymbol{\phi}\colon C^\infty(M,\mathbb{R}) \times \mathcal{R}\mathrm{iem}(M) \to [0,\infty]$ is a uniform quasi-flatzoomer independently of a choice of the metric $\eta$. If a metric $g$ is fixed, we write a quasiflatzoomer as a functional $\boldsymbol{\phi}\colon C^\infty(M,\mathbb{R}) \to [0,\infty]$.

**Remark 4.8.** In particular, a functional $\boldsymbol{\phi}\colon C^\infty(M,\mathbb{R}) \times \mathcal{R}\mathrm{iem}(M) \to [0,\infty]$ is a uniform quasiflatzoomer if there exist $k, d \in \mathbb{N}$, a continuous map $g \mapsto P_g \in C^0(M, \mathcal{P}^d_{k+1})$ and a continuous positive function $\alpha : g \mapsto \alpha(g) \in \mathbb{R}_{>0}$ on $\mathcal{R}\mathrm{iem}(M)$ such that

$$\boldsymbol{\phi}(u,g) \;\le\; ||\mathrm{e}^{-\alpha u(y)} P_g(y) \Big( u(y),\, |\nabla^1_\eta u|_\eta(y),\, \ldots,\, |\nabla^k_\eta u|_\eta(y) \Big) ||_{C^0(M)}$$

**Example 4.9.** Given a metric $g$ and a uniform flatzoomer $\boldsymbol{\Phi}\colon C^\infty(M,\mathbb{R}) \to C^0(M,\mathbb{R}_{\ge 0})$ of degree $k$, the functional

$$\boldsymbol{\phi} : u \mapsto \sup\{ \boldsymbol{\Phi}(u(x)) \mid x \in M \}$$

is a uniform quasi-flatzoomer of degree $k$. $\square$

Now let $(M,g)$ be a Riemannian manifold, and $\mathcal{K} = (K_l)_{l\in\mathbb{N}}$ be a compact exhaustion of $M$. We say that a *functional* $\boldsymbol{\phi}\colon C^\infty(M,\mathbb{R}) \to [0,\infty]$ *is a quasi-flatzoomer for the family* $\mathcal{K}$ if for each $l \in \mathbb{N}$ the functional

$$\boldsymbol{\phi}|_{K_l}\colon u \mapsto \boldsymbol{\phi}(u|_{K_l})$$

is a quasi-flatzoomer. Analogously to the proof of Lemma 4.6 (replacing pointwise estimates by taking maxima on $C$) one can show:

**Lemma 4.10.** *Let $m \in \mathbb{N}$. Assume each functional $\boldsymbol{\phi}_i\colon C^\infty(M,\mathbb{R}) \to [0,\infty]$ is a uniform quasi-flatzoomer of degree $k_i$ for the family $\mathcal{K}$, $i \in \{1,\ldots,m\}$. Assume also that $Q \in C^0(M \times (\mathbb{R}_{\ge 0})^m,\, \mathbb{R}_{\ge 0})$ is homogeneous-polynomially bounded. Then the functional $\boldsymbol{\phi}\colon C^\infty(M,\mathbb{R}) \to [0,\infty)$ defined by*

$$\boldsymbol{\phi}(u)(x) := Q(x, \boldsymbol{\phi}_1(u)(x), \ldots, \boldsymbol{\phi}_m(u)(x))$$

*is a uniform quasi-flatzoomer for the family $\mathcal{K}$, of degree $\le \max\{k_1, ..., k_m\}$.*

We denote by $\mathrm{inj}_g$ the injectivity radius of $(M,g)$. For $x \in M$ and $r > 0$, we denote by

$$B_r(x) = \{\ z \mid\ \mathrm{dist}_g(z,x) < r\ \} \subset M$$

an open ball centered at $x$. Recall the convexity radius $\mathrm{conv}_g$ is defined as

$$\mathrm{conv}_g(x) = \sup\{\ \rho \in [0, \mathrm{inj}_g] \mid \text{for every } r \in [0,\rho) \text{ the ball } B_r(x) \text{ is strongly } g\text{-convex}\}\,.$$

**Theorem 4.11.** *Let $g$ be a Riemannian metric on a manifold $M$. Then for any compact $K \subset M$, the functionals $\boldsymbol{\Phi}^{\mathrm{inj}}_K, \boldsymbol{\Phi}^{\mathrm{conv}}_K\colon C^\infty(M,\mathbb{R}) \to \mathbb{R}_{\ge 0}$ given by*

$$\boldsymbol{\Phi}^{\mathrm{inj}}_K(u) := \sup\{\ 1/\,\mathrm{inj}_{g[u]}(x) \mid x \in K\ \}, \quad \boldsymbol{\Phi}^{\mathrm{conv}}_K(u) := \sup\{\ 1/\,\mathrm{conv}_{g[u]}(x) \mid x \in K\ \}$$

*are uniform quasi-flatzoomers of degree 2.*

*Proof.* Let $\mathcal{A}$ be an atlas for $M$. We choose a (parametrized) finite cover $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ of $K$ by open sets $U_i$ each of which has compact closure contained in the domain of some $\mathcal{A}$-chart $\varphi_i$.
Let $n := \dim M$. For any $u \in C^\infty(M,\mathbb{R})$, we can consider the Christoffel symbols ${}^{g[u]}\Gamma^c_{ab}$ of the metric $g[u]$ with respect to the coordinates determined by $\varphi_i$. Since $U_i$ has compact closure in $\mathrm{dom}(\varphi_i)$, there exists a constant $A_i \in \mathbb{R}_{>0}$ $C^1$-continuously depending on $g$ such that

$$|{}^{g[u]}\Gamma^c_{ab}| \leq A_i(1 + |\mathrm{d}u|_g)$$

holds pointwise on $U_i$ for every $u \in C^\infty(M,\mathbb{R})$: we have

$$\begin{aligned} {}^{g[u]}\Gamma^c_{ab} &= \frac{1}{2}\sum_{m=1}^{n} g[u]^{cm} \left(\partial_a g[u]_{bm} + \partial_b g[u]_{am} - \partial_m g[u]_{ab}\right) \\ &= \frac{1}{2}\sum_{m=1}^{n} \frac{g^{cm}}{\mathrm{e}^{2u}} \left(\mathrm{e}^{2u}\left(\partial_a g_{bm} + \partial_b g_{am} - \partial_m g_{ab}\right) + 2\mathrm{e}^{2u}\left(\partial_a u\; g_{bm} + \partial_b u\; g_{am} - \partial_m u\; g_{ab}\right)\right) . \end{aligned}$$

For $i \in \mathbb{N}$, we denote the Euclidean metric on $TM_{\mathrm{dom}(\varphi_i)}$, obtained via $\varphi_i$-pullback by $\mathrm{eucl}_i$. There exists a constant $C_i \in \mathbb{R}_{>0}$ depending $C^0$-continuously on $g$ such that

$$C_i\,|v|_{\mathrm{eucl}_i} \geq |v|_g \geq C_i^{-1}\,|v|_{\mathrm{eucl}_i}$$

holds for every $x \in U_i \cap L$ and every $v \in T_xM$. We define $H_i := 4n^2 A_i C_i^3 \in \mathbb{R}_{>0}$.
Since $\mathcal{U}$ is finite and can be chosen the same for a $C^1$-small variation in the metric, we can put $H := \max\{H_i\} \in \mathbb{R}$.
Let $Q(x,s) = \frac{2}{\pi}s^{1/2}$, then Example 4.9 and Lemma 4.10 tell us the functional $\mathbf{\Phi}_0 \colon C^\infty(M,\mathbb{R}) \to \mathbb{R}_{\geq 0}$ given by

$$\mathbf{\Phi}_0(u) := \tfrac{2}{\pi}\left|\mathrm{Rm}_{g[u]}\right|^{1/2}_{C^0(g[u])}$$

is a uniform quasi-flatzoomer of degree 2. Moreover, $\mathbf{\Phi}_1 \colon C^\infty(M,\mathbb{R}) \to \mathbb{R}_{\geq 0}$ given by

$$\mathbf{\Phi}_1(u) := \left|\mathrm{e}^{-u} H \cdot (1 + |\mathrm{d}u|_g)\right|_{C^0}$$

is obviously a uniform quasi-flatzoomer of degree 1.
There exists a (sufficiently large) function $u_1 \in C^0(M,\mathbb{R})$ such that for every $i \in \mathbb{N}$ and for every $x \in M$, there is an index $j$ such that

$$B_1^{g[u_1]}(x) \subseteq U_j\,.$$

Clearly, $\mathbf{\Phi}_2 \colon C^\infty(M,\mathbb{R}) \to \mathbb{R}_{\geq 0}$ given by $\mathbf{\Phi}_2(u) := 4|\mathrm{e}^{-u}\mathrm{e}^{u_1}|_{C^0}$ is a uniform quasi-flatzoomer of degree 0.
By Lemma 4.6, $\mathbf{\Psi} := \mathbf{\Phi}_0 + \mathbf{\Phi}_1 + \mathbf{\Phi}_2$ is a uniform quasi-flatzoomer of degree 2.

**Lemma 4.12.** *Let $\mathbf{\Psi} := \mathbf{\Phi}_0 + \mathbf{\Phi}_1 + \mathbf{\Phi}_2$ be as above. Then the inequality*

$$1/\,\mathrm{inj}_{g[u]}(x) \;\leq\; 1/\,\mathrm{conv}_{g[u]}(x) \;\leq\; \mathbf{\Psi}(g,u) \tag{4.6}$$

*holds for all $i \in \mathbb{N}$ and $x \in K_i \setminus K_{i-1}$ and $u \in C^\infty(M,\mathbb{R})$ which satisfy $u > u_0$ on $K_{i+1} \setminus K_{i-2}$.*

We notice that Lemma 4.12 implies Theorem 4.11.

*Proof.* In order to verify (4.6), only the right-hand side inequality has to be checked. We denote by $B_r^g(x)$ a closed ball (with respect to $g$) centered at $x$ of radius $r$. According to [11, Corollary 3.5], it suffices to verify that for all indices $i$ and points $x \in (K_i \setminus K_{i-1})$ and $u \in C^\infty(M, \mathbb{R})$ which satisfy $u > u_0$, there exists an $r > 0$ such that the ball $B_r^{g[u]}(x)$ is compact and the following inequalities hold:

$$\frac{2}{\pi}\left|\max\left\{\sec_{g[u]}(\sigma) \;\middle|\; z \in B_r^{g[u]}(x),\ \sigma \in \mathrm{Gr}_2(T_zM)\right\}\right|^{1/2} \leq \mathbf{\Phi}_0(u,g)\,, \tag{4.7}$$

$$\sup\left\{4/\operatorname{length}(\gamma) \;\middle|\; \gamma \subset B_r^{g[u]}(x) \text{ is a noninjective } g[u]\text{-geodesic}\right\} \leq \mathbf{\Phi}_1(u,g)\,, \tag{4.8}$$

$$\frac{4}{r} \leq \mathbf{\Phi}_2(u,g)\,. \tag{4.9}$$

Here we assume the convention $\sup \emptyset := 0$. Now we show that $r := 1/\sup\left\{\mathrm{e}^{u_1(y)-u(y)} \;\middle|\; y \in M\right\}$ has these properties. First, it satisfies (4.9) tautologically. Moreover, with $q := \inf\left\{\mathrm{e}^{u(y)-u_1(y)} \;\middle|\; y \in M\right\}$ we obtain

$$B := B_r^{g[u]}(x) = B_r^{\exp(2u-2u_1)g[u_1]}(x) \subseteq B_r^{q^2 g[u_1]}(x) = B_{r/q}^{g[u_1]}(x) = B_1^{g[u_1]}(x) \subseteq U_j$$

for some $j \in \mathbb{N}$. The ball $B$ is a connected closed subset of $M$, and $B$ is contained in $U_j$, whose closure in $M$ is a compact subset of a chart domain. All this together implies that $B$ is compact. The inequality (4.7) holds, indeed, for each $z \in B$ and each $\sigma \in \mathrm{Gr}_2(T_zM)$, we choose a $g[u]$-orthonormal basis $(e_1, e_2)$ of $\sigma$. This yields

$$\left|\sec_{g[u]}(\sigma)\right| = \left|\mathrm{Rm}_{g[u]}(e_1, e_2, e_1, e_2)\right| \leq \left|\mathrm{Rm}_{g[u]}\right|_{g[u]}\,.$$

The definition of $\mathbf{\Phi}_0(u)$ implies (4.7).
It remains to check (4.8). Let $\gamma\colon [0,\ell] \to B$ be an arclength-parametrized $g[u]$-geodesic with $\gamma(0) = \gamma(\ell)$. There exists an $s_0 \in [0,\ell]$ with $u(\gamma(s_0)) = \min_{s\in[0,\ell]} u(\gamma(s))$.
Since $B \subseteq U_j \subseteq \mathrm{dom}(\varphi_j)$, the euclidean metric $\mathrm{eucl}_j$ is defined on $B$ and we can regard $B$ as a subset of the vector space $\mathbb{R}^n$. There is an $s_1 \in [0,\ell]$ with $\langle \gamma'(s_1), \gamma'(s_0)\rangle_{\mathrm{eucl}_j} \leq 0$, because the map $w\colon [0,\ell] \ni t \mapsto \langle \gamma'(t), \gamma'(s_0)\rangle_{\mathrm{eucl}_j}$ satisfies

$$\int_0^\ell w(t)\,\mathrm{d}t = \big\langle \gamma(\ell) - \gamma(0), \gamma'(s_0)\big\rangle_{\mathrm{eucl}_j} = 0.$$

In particular, we have $|\gamma'(s_0)|_{\mathrm{eucl}_j} \leq |\gamma'(s_1) - \gamma'(s_0)|_{\mathrm{eucl}_j}$.
Denoting the components (with respect to the chosen coordinates) of a vector $v \in T_xM$ with $x \in B$ by $v_1, \ldots, v_n$, we have the following estimates:

$$C_j\, |v|_{\mathrm{eucl}_j} \geq |v|_g \geq C_j^{-1}\, |v|_{\mathrm{eucl}_j}\,, \qquad n^{1/2}\, |v|_{\mathrm{eucl}_j} \geq \sum_{a=1}^n |v_a|\,.$$

In particular,

$$\forall s \in [0,\ell]\colon\ n^{1/2} C_j \mathrm{e}^{-u(\gamma(s))} = n^{1/2} C_j \mathrm{e}^{-u(\gamma(s))}\left|\gamma'(s)\right|_{g[u]} = n^{1/2} C_j \left|\gamma'(s)\right|_g \geq \sum_{a=1}^n \left|\gamma_a'(s)\right|\,.$$

Using this and $\forall c\colon\ |\partial_c|_{\mathrm{eucl}_j} = 1$ and the $g[u]$-geodesic equation

$$\forall s \in [0,\ell]\colon\ \gamma''(s) = \sum_{c=1}^{n} \gamma''_c(s)\ \partial_c\Big(\gamma(s)\Big) = -\sum_{a,b,c=1}^{n} {}^{g[u]}\Gamma^c_{ab}\Big(\gamma(s)\Big)\ \gamma'_a(s)\gamma'_b(s)\ \partial_c\Big(\gamma(s)\Big)\,,$$

in [11, proof of Theorem 3.8], one obtains (setting there $K_{i+1} := K_i := L := M$, $K_{i-1} := K_{i-2} = \emptyset$):

$$\begin{aligned}\frac{4}{\ell} &\le H_j \cdot \|\mathrm{e}^{-u}(1 + |\,\mathrm{d}u|_g)\|_{C^0(U_j)} \\ &\le \|H \cdot \mathrm{e}^{-u}(1 + |\,\mathrm{d}u|_g)\|_{C^0(U_j)} \\ &= \mathbf{\Phi}_1(u,g)\,.\end{aligned}$$

Hence (4.8) also holds. This completes the proof of Lemma 4.12. □

This concludes the proof of Theorem 4.11. □

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